ON THE SPECIALIZATION THEOREM FOR ABELIAN VARIETIES

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Abstract. In this note, we apply Moriwaki's arithmetic height functions to obtain an analogue of Silverman's Specialization Theorem for families of Abelian varieties over K, where K is any field finitely generated over $\mathbb{Q}$.

Let k be a number field and $\pi : \mathcal{A} \to \mathcal{B}$ be a proper flat morphism of smooth projective varieties over k such that the generic fiber $\mathcal{A}_\eta$ is an Abelian variety defined over $k(\mathcal{B})$. For almost all absolutely irreducible divisors D/k on $\mathcal{B}$, $\mathcal{A}_D := \mathcal{A} \times_{\mathcal{B}} D$ is also a flat family of Abelian varieties, and our goal is to compare the Mordell-Weil rank of $\mathcal{A}_\eta$ with the Mordell-Weil rank of the generic fiber of $\mathcal{A}_D$. In fact, if we fix a projective embedding of $\mathcal{A}$, $\mathcal{B}$ into $\mathbb{P}^N$, and an integer $M > 0$, then we can show that for all but finitely-many divisors D of degree less than M, the rank of the generic fiber of $\mathcal{A}_D$ is at least the rank of $\mathcal{A}_\eta$.

This result is a an amusing example of the height machine in action. The main issue is to rephrase the problem in terms of Abelian varieties over function fields, and define the "right" height functions; the proofs then follow verbatim as in [**?**].

Assume now that we also have a proper, flat morphism $f : \mathcal{B} \to \mathcal{X}$ with generic fiber a smooth, irreducible curve C defined over $K := k(\mathcal{X})$. Composition gives a flat morphism $g := f \circ \pi : \mathcal{A} \to \mathcal{X}$ whose generic fiber is a smooth, irreducible variety A defined over K, and by base extension we have a flat morphism $\rho : A \to C$, whose generic fiber is $\mathcal{A}_\eta$. Observe that in this setting, divisors $D \in \mathrm{Div}(\mathcal{B})$ such that $f(D) = \mathcal{X}$ correspond to points on C. The next proposition (based on a classical geometric argument, see [**?**, Proposition 5.1]) shows that, up to replacing $\mathcal{B}$ and $\mathcal{A}$ by birationally equivalent varieties, we can always reduce to this situation.

Proposition 0.1. *Let $\mathcal{V}$ be a smooth projective variety of dimension n defined over k. Then, there exist finitely many birationally equivalent varieties $\nu_i : \tilde{\mathcal{V}}_i \to \mathcal{V}$ and proper flat morphisms $f_i : \tilde{\mathcal{V}}_i \to \mathbb{P}^{n-1}$ such that:*

1. *The generic fiber of f_i is a smooth, irreducible curve C_i defined over $k(\mathbb{P}^{n-1})$.*
2. *Let $\mathcal{U}_i$ be an affine open dense subset of $\mathcal{V}$ such that ν_i is an isomorphism on $\mathcal{U}_i$, and let $\mathcal{U} := \cap \mathcal{U}_i$. Then for all divisors $D \in \mathrm{Div}(\mathcal{V})$ such that $D \cap \mathcal{U} \neq \emptyset$, there exists an i such that $f_i(D') = \mathbb{P}^{n-1}$, where $D' := \overline{\nu_i^{-1}(D \cap \mathcal{U}_i)}$.*

Proof. Consider an embedding $\mathcal{V} \hookrightarrow \mathbb{P}^N$, and take a linear projection inducing a finite morphism $\phi : \mathcal{V} \to \mathbb{P}^n$. Let $p_0 \in \mathbb{P}^n$ be a point outside the ramification locus of ϕ, let $\mu_0 : \tilde{\mathbb{P}}_0 \to \mathbb{P}^n$ be the blow-up at p_0, and $\mu_0' : \tilde{\mathbb{P}}_0 \to \mathbb{P}^{n-1}$ be the associated morphism. If $\nu_0 : \tilde{\mathcal{V}}_0 \to \mathcal{V}$ is the blow-up of $\mathcal{V}$ at the points $\phi^{-1}(p_0)$, then we have a morphism $\tilde{\phi}_0 : \tilde{\mathcal{V}}_0 \to \tilde{\mathbb{P}}_0$ which, when composed with μ_0', gives a fibration in curves $f_0 := \mu_0' \circ \tilde{\phi}_0 : \tilde{\mathcal{V}}_0 \to \mathbb{P}^{n-1}$.

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Next, let D' be the proper transform of D under the blow-up μ_0. Then $f_0(D') \neq \mathbb{P}^{n-1}$ if and only if D is a cone with vertex $V_D \ni p_0$. Since the vertex of a cone is a linear subspace, the highest dimensional vertex is a hyperplane. Therefore, if we choose $n+1$ points $p_0, \dots p_n$ not all lying in a hyperplane, we see that for all divisors D in $\mathbb{P}^n$, we can find a morphism f_i such that $f_i(D') = \mathbb{P}^{n-1}$. Since ϕ is a finite morphism, the same holds for divisors in $\mathcal{V}$. $\diamondsuit$

The geometric proposition above allows us to reduce our problem to the following situation: Let K be a field finitely generated over $\mathbb{Q}$, and let C be a smooth projective curve defined over K. Suppose A is a smooth projective variety equipped with a proper flat morphism $\rho : A \to C$, such that the generic fiber A_ρ is an Abelian variety with Chow trace (τ, B). Let $x \in C(\overline{K})$ be a point for which the fiber A_x is nonsingular. Then the specialization map

$$\sigma_x : A(C/K) \to A_x(\overline{K})$$

is a homomorphism from the group of sections, to the group of points on the fiber.

Theorem 0.1. *Let Γ be a finitely generated free subgroup of $A(C/K)$ which injects in $A(C/K)/\tau B(K)$. Then the set*

$$\left\{x \in C(\overline{K}) : \sigma_x \text{ is not injective on } \Gamma\right\}$$

is a set of bounded height in $C(\overline{K})$. In particular, if $d \geq 1$, then σ_x is injective for all but finitely many $x \in \cup_{[L:K]\leq d} C(L)$. Furthermore, if A_ρ has trivial Chow trace, this shows that, excluding a finite number of points $x \in C$, the Mordell-Weil rank of the special fibers A_x is at least that of the generic fiber A_ρ.

The proof of this theorem is based on being able to measure the variation of certain height functions in a family of Abelian varieties. We briefly describe the necessary height functions below, before outlining the major steps in the proof.

Arithmetic Height. We consider the arithmetic height functions on K introduced by Moriwaki [**?**], and, as much as possible, stick to Moriwaki's notation and terminology. Let Z be a normal projective arithmetic variety whose function field is K, and fix nef C^∞-hermitian line bundles $\overline{H}_1, \overline{H}_2, ..., \overline{H}_d$ on Z. The collection $(Z; \overline{H}_1, \overline{H}_2, ..., \overline{H}_d)$ is called a polarization of Z, and will be denoted by $\overline{Z}$.

Now suppose that X is a smooth projective variety over K, and L a line bundle on X. If $\mathcal{X}$ is a projective arithmetic variety over Z and $\overline{\mathcal{L}}$ a hermitian line bundle on $\mathcal{X}$ with $\mathcal{X}_K = X$ and $\mathcal{L}_K = L$ then the pair $(\mathcal{X}, \overline{\mathcal{L}})$ is a model for (X, L), and we can define a height function using Arakelov intersection theory, as follows: $h^{\overline{Z}}_{(\mathcal{X},\overline{\mathcal{L}})} : X(\overline{K}) \to \mathbb{R}$ via

$$h^{\overline{Z}}_{(\mathcal{X},\overline{\mathcal{L}})}(P) := \frac{\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}\big|_{\Delta_P}) \cdot \widehat{c}_1(f^*\overline{H}_1\big|_{\Delta_P}) \cdots \widehat{c}_1(f^*\overline{H}_d\big|_{\Delta_P})\right)}{[K(P):K]},$$

where Δ_P is the Zariski closure of the point P in $\mathrm{Spec}(\overline{K}) \xrightarrow{P} X \hookrightarrow \mathcal{X}$, and $f : \mathcal{X} \to Z$ is the canonical morphism. Furthermore, if the polarization Z is big (as defined in [**?**, Section 2]), then $h^{\overline{Z}}_{(\mathcal{X},\overline{\mathcal{L}})}$ is an arithmetic height function, and denoted by $h^{\mathrm{arith}}_{(\mathcal{X},\overline{\mathcal{L}})}$. In fact, [**?**, Corollary 3.3.5] shows that, up to a bounded function, the height $h^{\overline{Z}}_{(\mathcal{X},\overline{\mathcal{L}})}$ does not depend on the choice of model $(\mathcal{X}, \overline{\mathcal{L}})$ for (X, L), and hence we will denote the arithmetic height function simply by $h^{\mathrm{arith}}_{(X,L)}$, or h^{arith}_L when X is understood.

Proposition 0.2 (Height Machine)**.** *Let X be a smooth projective variety over K, and L a line bundle on X. Then the arithmetic height function h_L^{arith} satisfies the following properties:*

(1) *If M is another line bundle on X, then $h_{L\otimes M}^{\text{arith}} = h_L^{\text{arith}} + h_M^{\text{arith}} + O(1)$, while $h_{L^{\otimes -1}}^{\text{arith}} = -h_L^{\text{arith}} + O(1)$.*

(2) *Let $Bs(L)$ denote the base locus of L, and set $SBs(L) := \cap_{n>0} Bs(L^{\otimes n})$. Then h_L^{arith} is bounded below on $(X \backslash SBs(L))(\overline{K})$, and in particular:*
   (a) *If L is ample, then h_L^{arith} is bounded below.*
   (b) *If $L = \mathcal{O}_X$ then $h_L^{\text{arith}} = O(1)$.*

(3) *If N is any number, and R a positive integer, then the set*
$$\left\{ P \in X(\overline{K}) | h_L^{\text{arith}}(P) \leq N \text{ and } [K(P):K] \leq R \right\}$$
is finite.

(4) *Let $q : X \to Y$ be a morphism of smooth, projective varieties over K, and M a line bundle on Y; then*
$$h_{(X,q^*(M))}^{\text{arith}}(P) = h_{(Y,M)}^{\text{arith}}(q(P)) \qquad \text{for all } P \in X(\overline{K}).$$

(5) *If L is ample, and M is any line bundle on X, then there is a constant c such that*
$$h_M^{\text{arith}} < c h_L^{\text{arith}} + O(1).$$

(6) *Suppose X is a smooth projective curve over K, and if D is a divisor write h_D^{arith} for $h_{L(D)}^{\text{arith}}$, where $L(D)$ is the line bundle associated to D. If D is of degree $d > 0$, and E is a divisor of degree e, then*
$$\lim_{h_D^{\text{arith}}(t) \to \infty} \frac{h_E^{\text{arith}}(t)}{h_D^{\text{arith}}(t)} = \frac{e}{d}.$$

Proof. Properties (1)-(3) are [**?**, Proposition 3.3.7], while Property (4) is [**?**, Theorem 4.3]. To prove Property (5), we start with the definition of arithmetic height, and note that if $f : Y \to B$ is the canonical morphism on Y, then the canonical morphism on X is given by $f' = f \circ q$:

$$\begin{aligned}
h_{(X,q^*(M))}(P) &:= \frac{\widehat{\deg}\left((f'^*)\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\right)\cdot\widehat{c}_1(q^*(\bar{M}))\cdot(\Delta_P,0)\right)}{[K(P):K]} \\
&= \frac{\widehat{\deg}\left((q^*\circ f^*)\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\right)\cdot q^*\widehat{c}_1(\bar{M})\cdot(\Delta_P,0)\right)}{[K(P):K]} \\
&= \frac{\deg(\Delta_P\to\Delta_{q(P)})\widehat{\deg}\left((f^*)\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\right)\cdot\widehat{c}_1(\bar{M})\cdot(\Delta_{q(P)},0)\right)}{[K(P):K]} \\
&= h_{(Y,M)}(q(P)),
\end{aligned}$$

where the third equation follows from the Projection formula ([**?**, Proposition 1.3]), and the final step is again by definition.

To prove Property (6), we note that there is some constant a such that $L^{\otimes a} \otimes M^{\otimes -1}$ is ample. Hence, by Property (3.a) we have $h_{L^{\otimes a} \otimes M^{\otimes -1}} > O(1)$. The result now follows by Properties (1) and (2).

Finally, the proof of Property (7) is very similar to the proof for heights over number fields (see for example [**?**, Chapter 4, Corollary 3.5]). $\diamondsuit$

If A/K is an Abelian variety, then we can also define a canonical height $\hat{h}_L^{\mathrm{arith}}$, which has the following properties:

Proposition 0.3 (Canonical Height Machine). (1) *If M is another line bundle on A, then $\hat{h}_{L\otimes M}^{\mathrm{arith}} = \hat{h}_L^{\mathrm{arith}} + \hat{h}_M^{\mathrm{arith}}$, while $\hat{h}_{L^{\otimes -1}}^{\mathrm{arith}} = -\hat{h}_L^{\mathrm{arith}}$.*
If we assume furthermore that L is ample, then we have:

(3) *There is a constant c, such that if M is another line bundle on A, then $\hat{h}_M^{\mathrm{arith}} \leq c\hat{h}_L^{\mathrm{arith}}$.*
(4) *$\hat{h}_L^{\mathrm{arith}}(x) \geq 0$ for all $x \in A(\overline{K})$, and equality holds if and only if x is a torsion point.*

Proof. This is [**?**, Propositions 3.4.1 and 3.4.2]. He proves these under the condition that L is a symmetric line bundle, but the proofs hold verbatim for L anti-symmetric, hence for all line bundles L. $\diamondsuit$

Geometric Height. Since K is a global field, and C/K is a smooth projective curve, $K(C)$ is also a global field, and we can define the geometric height $h_{K(C)}^{\mathrm{geom}}$ so that, for any $x \in K(C)$, $h_{K(C)}^{\mathrm{geom}}([x,1])$ is the degree of the morphism $[x,1] : C \to \mathbb{P}^1$. If A is an Abelian variety defined over K, then we also have a canonical height $\hat{h}_{K(C)}^{\mathrm{geom}}$, which is non-degenerate on $A(\overline{K})/(\tau B(K) + A_{\mathrm{tors}})$ [**?**, Chapter 6, Theorem 5.4]. Both $h_{K(C)}^{\mathrm{geom}}$ and $\hat{h}_{K(C)}^{\mathrm{geom}}$ satisfy the usual properties of heights/canonical heights, see [**?**, Chapter 3].

Equipped with these height functions, the proof of the following theorems then proceeds exactly as in [**?**], and leads to a proof of Theorem 0.1:

Fix a line bundle L on A, and let $h_{(A,L)}^{\mathrm{arith}}$ be the associated arithmetic height function. For each $t \in C(\overline{K})$ let L_t be its restriction to A_t, and D_ρ be its restriction to the generic fiber A_ρ. Let $C^0 \subset C$ be an affine open subset such that for all $t \in C^0(\overline{K})$, the fiber A_t is an Abelian variety. Then there is a geometric canonical height

$$\hat{h}_{(A_\rho,D_\rho)}^{\mathrm{geom}} : A_\rho(K(C)) \to \mathbb{R},$$

and for each $t \in C^0(\overline{K})$, an arithmetic canonical height

$$\hat{h}_{(A_t,L_t)}^{\mathrm{arith}} : A_t(\overline{K}) \to \mathbb{R}.$$

Set $U := \pi^{-1}(C^0)$. The canonical heights $\hat{h}_{(A_t,L_t)}^{\mathrm{arith}}$ on the good fibers A_t can be fitted together to give a "canonical height" on $U(\overline{K})$, $\hat{h}_L : U(\overline{K}) \to \mathbb{R}$. If we fix also a line bundle M on C, then

Theorem 0.2. *There is a constant c, depending on M, L, and the family $A \to C$, such that*

$$\left|\hat{h}_L(P) - h_L^{\mathrm{arith}}(P)\right| < ch_M^{\mathrm{arith}}(P) + O(1) \qquad \text{for all } P \in U(\overline{K}).$$

(The $O(1)$ depends on the choice of heights h_L^{arith} and h_M^{arith} but not on P.)

Consider now the case where C is a smooth projective curve over K, and fix an arithmetic height on C as follows: Let D be a divisor on C with $\deg D > 0$, and define

$$h_C^{\mathrm{arith}} := \frac{1}{\deg D} h_{(C,D)}^{\mathrm{arith}}.$$

Theorem 0.3. *With notation as above, fix a section $P \in A(C)$. Then*

$$\lim_{\substack{t \in C^0(\overline{K}) \\ h_C^{\mathrm{arith}}(t) \to \infty}} \frac{\hat{h}^{\mathrm{arith}}_{(A_t, L_t)}(P_t)}{h_C^{\mathrm{arith}}(t)} = \hat{h}^{\mathrm{geom}}_{(A_\rho, D_\rho)}(P_\rho).$$

Note that by Property (7) of arithmetic heights, this result does not depend on the choice of arithmetic height on C.

Finally, we observe that in the case that A is an elliptic surface, another crank of the height machine yields the following sharpened estimate á la Tate [**?**] for Theorem 0.3:

Theorem 0.4. *Assume that $\pi : A \to C$ is an elliptic surface, and assume that $P \in A(C/K)$ is a section. Then*

$$\hat{h}^{\mathrm{arith}}_{(A_t, L_t)}(P_t) = \hat{h}^{\mathrm{geom}}_{(A_\rho, D_\rho)}(P_\rho) h_C^{\mathrm{arith}}(t) + O\left(\sqrt{h_C^{\mathrm{arith}}(t)} + 1\right)$$

Tate's proof requires the existence of a good compactification of the Neron model, hence is not known to apply to families of higher dimensional Abelian varieties. However, in the case of number fields, Call[**?**, Theorem I], using a different approach, generalized the result to the case of a one-parameter family of Abelian varieties. It remains an interesting question whether an analogous estimate exists for a family of Abelian varieties over K.

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